# ON DUAL AUTOMORPHISM-INVARIANT MODULES

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1. Generalized Notions of Injectivity and Projectivity

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4. Main Results
    - Dual automorphism $N$-invariant modules
    - S-ADS Modules

- A module $M$ is called quasi-injective, (or self-injective) if for every submodule $N$ of $M$ every $R$-homomorphism of $N$ into $M$ can be extended to $R$-endomorphism of $M$(Johnson and Wong, J. London Math. Soc. 1961). It is a generalization of injectivity.

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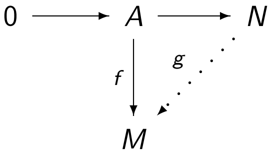
- A module $M$ is called quasi-projective if for any epimorphism $g : M \to M/T$ and any morphism $f : M \to M/T$ there exists a homomorphism $h : M \to M$ such that $f = gh$ (Y. Miyashita, 1966). It is a generalization of projectivity.

- A module $M$ was called $N$-pseudo- injective if for any submodule $A$ of $N$ every monomorphism $f : A \to M$ can be extended to $g : N \to M$.

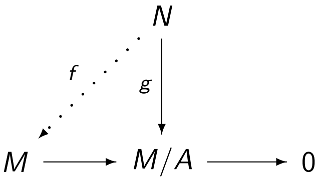


- A module $M$ was called $N$-pseudo- injective if for any submodule $A$ of $N$ every monomorphism $f : A \to M$ can be extended to $g : N \to M$.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & N \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ & & M & & \end{array}$$

- A module $M$ was called pseudo- injective by Jain and Singh, (J. Math. Sci., 1967) if it is $M$-pseudo- injective.

- A module $M$ was called $N$-pseudo- projective if for every submodule $A$ of $M$ and any epimorphism $g : N \to M/A$ can be lifted to a homomorphism $f : N \to M$. If



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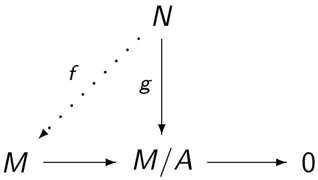


- $M$ is $M$-pseudo- projective then it is called pseudo- projective (Bican, Acta Mathematica Academiae Scientiarum Hungaricae, 1976).

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- A module $M$ which is invariant under automorphisms of its injective envelope has been called an *automorphism invariant* module by Lee and Zhou equivalently $M$ is automorphism-invariant if every isomorphism between two essential submodules of $M$ extends to an automorphism of $M$ (J. Alg. Appl., (2013)) (for modules over any ring).

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- Quasi-injective and pseudo-injective modules modules are automorphism invariant (by Lee and Zhou, J. Algebra Appl., 2013).

- Pseudo-injective modules and automorphism-invariant modules coincide (by Er, Singh and Srivastava, J. Algebra, 2013 ).

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- $M$ is automorphism $N$-invariant if for any essential submodule $A$ of $N$, any essential monomorphism $f : A \to M$ can be extended to some $g \in Hom(N, M)$ (Quynh and Kosan, J. Alg. App., 2015).

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- $M$ is called automorphism-invariant if $M$ is automorphism $M$-invariant.
- If $M$ is pseudo-$N$-injective then $M$ is automorphism $N$-invariant.

- A right $R$-module $M$ is called *dual automorphism-invariant* if whenever $K_1$ and $K_2$ are small submodules of $M$, then any epimorphism $\eta : M/K_1 \to M/K_2$ with small kernel lifts to an endomorphism $\varphi$ of $M$ (Singh and Srivastava, J. Alg., 2013)

$$\begin{array}{ccc} M & \xrightarrow[\varphi]{} & M \\ \downarrow & & \downarrow \\ M/K_1 & \xrightarrow[\eta]{} & M/K_2 \end{array}$$

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- Any pseudo-projective and quasi-projective modules are dual automorphism-invariant (by Singh and Srivastava).

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- Converse is true over right perfect rings (by Guil Asensio, P. A., Keskin Tutuncu, D., Kalebogaz, B., Srivastava, A. K.)

# Layout

Rings are associative with unity and modules are unital right $R$-modules.

Rings are associative with unity and modules are unital right $R$-modules. The purpose of this paper is to initiate the study of dual automorphism $N$-invariant modules via $N$-pseudo-projective modules.

### Definition

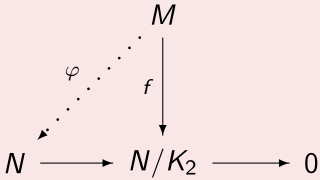
We call $M$ *dual automorphism $N$-invariant* if, whenever $K_1$ is a small submodule of $M$ and $K_2$ is a small submodule of $N$, then any epimorphism $p : M/K_1 \to N/K_2$ with small kernel lifts to a homomorphism $\varphi : M \to N$. That is:

$$\begin{array}{ccc} M & \xrightarrow[\varphi]{} & N \\ \downarrow & & \downarrow \\ M/K_1 & \xrightarrow[\eta]{} & N/K_2 \end{array}$$

## Theorem (Ş., Quynh)

The following conditions are equivalent for a right $R$-module $M$:

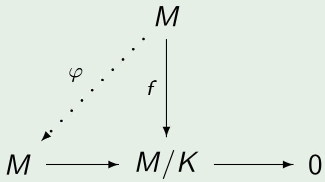
1. $M$ is dual automorphism $N$-invariant.
2. For any small submodule $K_1$ of $M$ and small submodule $K_2$ of $N$, every epimorphism $p : M/K_1 \to N/K_2$ with small kernel lifts to an epimorphism $\varphi : M \to N$.
3. For any small submodule $K_2$ of $N$, every epimorphism $f : M \to N/K_2$ with small kernel lifts to a homomorphism $\varphi : M \to N$.



## Corollary ( Ş., Quynh)

The following conditions are equivalent for a right $R$-module $M$:

1. $M$ is dual automorphism invariant.
2. For any small submodule $K$ of $M$, every epimorphism $f : M \to M/K$ with small kernel lifts to an endomorphism of $M$.



**Recall:** Let $N$ and $L$ be submodules of $M$. The module $N$ is called a *supplement* of $L$ in $M$ if $M = N + L$ and $N \cap L \ll N$. $M$ is called *supplemented* if every submodule of $M$ has a supplement in $M$.

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**Proposition ( Ş., Quynh)**

Let $M$ and $N$ be modules and $X = M \oplus N$. The following conditions are equivalent:

1. $M$ is dual automorphism $N$-invariant.
2. For each submodule $K$ of $X$ such that $N$ is a supplement of $K$ in $X$, there exists $C \leq K$ such that $N \oplus C = X$.

A module $M$ is called a *hollow* module if every proper submodule of $M$ is small in $M$. The following observation was proved for local modules by Singh and Srivastava (Journal of Algebra,2012).

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**Proposition ( Ş., Quynh)**

Assume that $M_1, M_2$ are two hollow modules. If $M_1$ is dual automorphism $M_2$-invariant, then $M_1$ is $M_2$-projective.

**Theorem (Ş., Quynh)**

Let $M$ and $N$ be two $R$-modules.

1. Every direct summand of a dual automorphism $M$-invariant module is also dual automorphism $M$-invariant.
2. $M$ is dual automorphism $N$-invariant if and only if any isomorphism $f : M/B \to N/A$ with $B \ll M$ and $A \ll N$ lifts to a homomorphism from $M$ to $N$.
3. If $M$ is a dual automorphism $N$-invariant module and $K \cong N$, then $M$ is dual automorphism $K$-invariant.
4. Assume that $N = A \oplus B$ and $M = C \oplus D$ such that there exists a small epimorphism from $D$ to $B$. If $M$ is dual automorphism $N$-invariant, then $C$ is dual automorphism $A$-invariant.

The following theorem extends Singh and Srivastava (Journal of Algebra,2012).

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**Theorem (Ş., Quynh)**

Let $\pi_1 : P_1 \to M$ and $\pi_2 : P_2 \to N$ be projective covers. Then the following conditions are equivalent.

1. $M$ is dual automorphism $N$-invariant.
2. $\sigma(Ker(\pi_1)) \leq Ker(\pi_2)$ for any isomorphism $\sigma : P_1 \to P_2$.

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**Corollary ( Ş., Quynh)**

Let $\pi : P \to M$ be a projective cover. Then $M$ is dual automorphism-invariant if and only if $\sigma(Ker(\pi)) \leq Ker(\pi)$ for any isomorphism $\sigma : P \to P$.

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**Theorem (Ş., Quynh)**

Let $M$ and $N$ be mutually dual automorphism invariant modules and $\pi_1 : P_1 \to M$ and $\pi_2 : P_2 \to N$ be projective covers. If $P_1 \cong P_2$, then every isomorphism $\sigma : P_1 \to P_2$ reduces an isomorphism from $Ker(\pi_1)$ to $Ker(\pi_2)$.

# Layout

- A right module $M$ over a ring $R$ is said to be ADS if for every decomposition $M = S \oplus T$ and every complement $T'$ of $S$, we have $M = S \oplus T'$. (see, Fuchs, Infinite Abelian Groups, 1970)

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- An $R$-module $M$ is ADS if and only if for each decomposition $M = S \oplus T$, $S$ and $T$ are mutually injective.
- A module $M$ is called an *e-ADS module* if, for every decomposition $M = S \oplus T$ and every complement $T'$ of $S$ with $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq^e S$, we have $M = S \oplus T'$ (Kosan and Quynh).

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- $M$ is an e-ADS module if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively automorphism invariant .

Any module $M$ is called *amply supplemented* if $B$ contains a supplement of $A$ in $M$ whenever $M = A + B$.

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**Theorem (Ş., Quynh)**

Assume that an amply supplemented $R$-module $X$ has a decomposition $X = M \oplus N$ for some $R$-modules $M$ and $N$. Then the following conditions are equivalent:

1. $M$ is dual automorphism $N$-invariant.
2. For any supplement $K$ of $N$ in $X$ with $K + M = X$ and $(K \cap M) \ll X$ , the module $X$ has a decomposition $X = K \oplus N$.
3. For each submodule $K$ of $X$ such that $K$ is a supplement of $N$ in $X$ and $M$ is a supplement of $K$ in $X$, we have $X = K \oplus N$.

We call $M$ an *s-ADS-module* if for every decomposition $M = S \oplus T$ of $M$ and every supplement $T'$ of $S$ with $T' + T = M$ and $(T \cap T') \ll M$, we have $M = S \oplus T'$.

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**Theorem (Ş., Quynh)**

The following conditions are equivalent for a module $M$:

1. M is s-ADS.
2. For every decomposition $M = S \oplus T$, if $T'$ is supplement of $S$ in $M$ and $T$ is supplement of $T'$ in $M$, then $M = S \oplus T'$.

**Theorem (Ş., Quynh)**

An amply supplemented $R$-module $M$ is s-ADS if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively dual automorphism invariant.

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An amply supplemented $R$-module $M$ is s-ADS if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively dual automorphism invariant.

**Corollary ( Ş., Quynh)**

Every amply supplemented dual automorphism-invariant module is s-ADS.

A right $R$-module $M$ is said to be $ADS^*$ if for every decomposition $M = S \oplus T$ and for every supplement $T'$ of $S$, we have $M = S \oplus T'$ (see Keskin, Bull. of Math. Sciences 2012). Clearly every $ADS^*$ module is s-ADS.

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**Theorem (Ş., Quynh)**

The following conditions are equivalent for a ring $R$:

1. $R$ is a right V-ring.
2. Every 2-generated right $R$-module is $ADS^*$.
3. Every 2-generated right $R$-module is s-ADS.

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